Worcester County Mathematics League

Varsity Meet 3
January 30, 2013

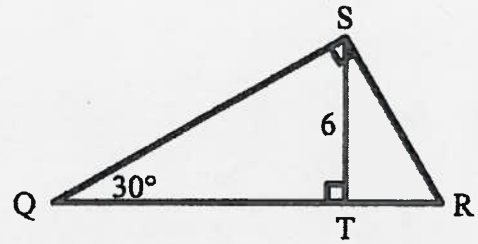
COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Varsity Meet 3 – January 30, 2013
Round 1: Similarity and Pythagorean Theorem

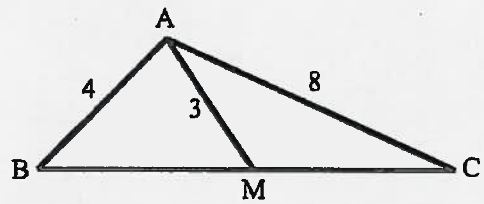
All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

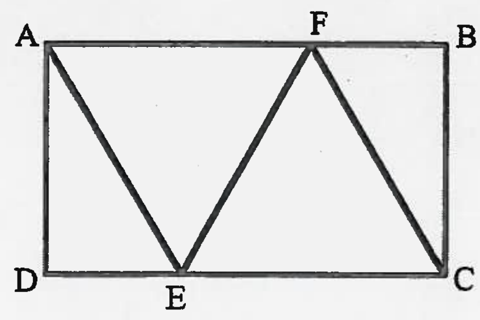
1. If $ST = 6$ and $m\angle Q = 30°$, find the length of QR.



2. Given that $AB = 4$, $AC = 8$, M is the midpoint of BC, and $AM = 3$, find BC.



3. $ABCD$ is a rectangle, and points E and F are chosen on sides $\overline{CD}$ and $\overline{AB}$ such that $AFCE$ is a rhombus. If $\overline{AB}$ has length a and $\overline{BC}$ has length b, find the length of $\overline{EF}$ in terms of a and b.



ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

Varsity Meet 3 – January 30, 2013
Round 2: Algebra I

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. At WOCOMAL College, 99% of the 100 students are girls, but only 98% of the students living on campus are girls. How many students live off campus?

2. Given that $\frac{1}{x}+\frac{1}{y}=\frac{3}{4}$ and $x^2y+xy^2=48$, find the sum of all possible positive values of x.

3. Given that $y^2+3x^2y^2=30x^2+517$ for integers x and y, find $3x^2y^2$.

ANSWERS

(1 pt.) 1. ________________ students

(2 pts.) 2. ________________

(3 pts.) 3. ________________

Varsity Meet 3 – January 30, 2013
Round 3: Functions

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If $f(x) = \dfrac{2}{x}$, find $f^{-1}(x)$.

2. If $f(x) = x^2 - x + 9$ and $g(x) = 3x + 2$, find all values of x such that $f(g(x)) = g(f(x))$.

3. Find the polynomial function $f(x)$ of least degree such that:
   - all coefficients of f are integers
   - all roots of f are integers
   - $f(0) = -1$
   - $f(3) = 128$.

   Write $f(x)$ as a polynomial in standard form.

ANSWERS

(1 pt.) 1. $f^{-1}(x) =$ ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. $f(x) =$ ______________________

Varsity Meet 3 – January 30, 2013
Round 4: Combinatorics

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Jane, Elizabeth, Mary, Catherine, and Lydia are seated at a round table. Lydia has disgraced the family, so Elizabeth refuses to sit next to her. How many distinguishable arrangements are possible? (All seats around the table are indistinguishable.)

2. Find all values of x such that

$$_{x^2}C_{5x-13} = {}_{x^2}C_{8x-27}.$$

(Here $_nC_r$, sometimes also written $\binom{n}{r}$, denotes the number of ways that r objects can be chosen from a set of n objects where order does not matter; i.e. *combinations*.)

3. On an idealized watch, the position of the minute hand is uniquely determined by the position of the hour hand. Therefore, if we position an hour hand and a minute hand on a watch in an arbitrary way, they may or may not correctly describe a point in time.

At how many points in a 12 hour period can the two hands be interchanged and still correctly describe a moment in time?

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

Varsity Meet 3 – January 30, 2013
Round 5: Analytic Geometry

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. What is the area of an ellipse with foci $(12, 5)$ and $(8, 15)$ and passing through the origin?

2. Calculate the shortest distance from the point $(2, -3)$ to the line given by the equation $y = \frac{3}{4}x + \frac{1}{4}$.

3. A non-vertical line with y-intercept 2 is tangent to the curve given by the equation $x^2 + y^2 = 12x + 20y - 100$. Find the slope of the line.

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

Varsity Meet 3 – January 30, 2013
TEAM ROUND

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (2 POINTS EACH)

APPROVED CALCULATORS ALLOWED

1. A store with no merchandise over \$100 had an unusual sale. It reduced the price of each item by a percentage equal to the price of the item in dollars. What is the maximum possible sale price of an item? Answer in dollars and cents, to the nearest cent.

2. Two poles, 100 feet high and 60 feet high, are 200 feet apart. A third pole 30 feet high is placed between so that all three poles are coplanar. If the distance from the top of each of the two poles to the top of the 30-foot pole is the same, how far (in feet) is the 30-foot pole from the 100-foot pole?

3. For Halloween, a candy company makes a special package of candy. How many combinations are possible if each bag contains 10 pieces of candy chosen from 5 flavors: strawberry, orange, grape, lemon, and apple?

4. Given that $f(x) = \sin x$, $g(x) = x^2 - 10x + 3$, $h(x) = 1/x$, and $j(x) = x - 7$, what composition of those functions, when applied to x, results in
$$\frac{1}{\sin^2(x-7) - 10\sin(x-7) + 3} - 7?$$

5. What is the equation of the line containing all points equidistant from the points $(6, 4)$ and $(-4, -2)$? Write your answer as $y = mx + b$ (slope-intercept form).

6. If
$$\sum_{n=3}^{15}\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) = a + bi$$
for a, b real numbers, find the ordered pair (a, b).

7. If r and s are the roots of the equation $ax^2 + bx + c = 0$, express $\frac{1}{r^2} + \frac{1}{s^2}$ in terms of a, b, and c.

8. Bowl A contains 7 red chips and 3 blue chips. Five of these are selected at random without replacement and put in Bowl B, which was originally empty. One chip is then drawn from Bowl B. Given that this chip is blue, find the conditional probability that 3 red chips and 2 blue chips were transferred from Bowl A to Bowl B. Express your answer as a fraction.

9. Reduce to a simplified fraction:
$$\frac{\dfrac{0.\overline{3} - 0.3\overline{9}}{0.\overline{3} + 0.\overline{4}}}{3.\overline{9}}$$

Varsity Meet 3 – January 30, 2013
TEAM ROUND ANSWER SHEET

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (2 POINTS EACH)

1. $ ________ . ________

2. ________________ feet

3. ________________

4. ________________

5. ________________

6. (________ , ________)

7. ________________

8. ________________

9. ________________

Varsity Meet 3 – January 30, 2013
ANSWERS

ROUND 1
(Bromfield, Shepherd Hill, †)

1. $8\sqrt{3}$
2. $2\sqrt{31}$
3. $\frac{b}{a}\sqrt{a^2+b^2}$

ROUND 2
(Tahanto, Bromfield, Mass Academy)

1. 50 students
2. $3+\sqrt{17}$
3. 588

ROUND 3
(South, Hudson, ‡)

1. $2/x$
2. $-3, 1$ (need both, either order)
3. x^4+2x^3-2x-1

ROUND 4
(QSC, Auburn, ‡)

1. 12
2. $5, 8$ (need both, either order)
3. 143

ROUND 5
(QSC, Assabet Valley, Worc Academy)

1. 210π
2. 19/5
3. 7/24

TEAM ROUND
(Mass Acad, Assabet Valley, Auburn, [unknown school], Tahanto, Worc Acad, Bromfield, Hudson, Quaboag)

1. $25.00
2. 90 feet
3. 1001
4. $j(h(g(f(j(x)))))$ or $(j \circ h \circ g \circ f \circ j)(x)$
5. $y = -\frac{5}{3}x + \frac{8}{3}$
6. $(-1, 0)$
7. $\frac{b^2-2ac}{c^2}$
8. 5/9
9. $-3/140$

† Posamentier and Salkind, *Challenging Problems in Geometry.*
‡ Edgar Dobriban, Princeton University '12.

Varsity Meet 3 – January 30, 2013
FULL SOLUTIONS

ROUND 1

1. By AA, all three triangles in the figure are 30-60-90 triangles with side lengths in the ratio $1 : \sqrt{3} : 2$. Therefore, $QS = 12$, so $SR = 12/\sqrt{3}$ and $QR = 24/\sqrt{3} = \boxed{8\sqrt{3}}$.

2. Draw an altitude down from A to meet $\overline{BC}$ at D. Let the length of the altitude be h, $MC = x$, $BD = a$, and $DM = x - a$. Then, using the Pythagorean Theorem on the three right triangles, we have that

$$\begin{aligned} a^2 + h^2 &= 16 \\ (x-a)^2 + h^2 &= 9 \\ (2x-a)^2 + h^2 &= 64 \end{aligned}$$

Expand the second two equations and subtract the first from each of them, obtaining

$$\begin{aligned} x^2 - 2ax &= -7 \\ 4x^2 - 4ax &= 48 \end{aligned}$$

Therefore, we find that $2x^2 = 62$, so $x = \sqrt{31}$. The question asks for the length of $\overline{BC}$, which is $2x$. Therefore, our answer is $\boxed{2\sqrt{31}}$.

3. Let the side length of the rhombus be r. Then, $AF = FC = r$, $FB = a - r$, and $BC = b$. Use the Pythagorean Theorem on triangle BCF to get $r = \dfrac{a^2 + b^2}{2a}$.

Next, drop a perpendicular from F, meeting $\overline{DC}$ at H. We have $DE = HC = a - r$, so $EH = 2r - a$. Plugging in $r = \dfrac{a^2+b^2}{2a}$, we have $EH = \dfrac{b^2}{a}$.

Finally, use the Pythagorean Theorem on triangle EFH to get $EF = \boxed{\dfrac{b}{a}\sqrt{a^2+b^2}}$.

ROUND 2

1. The are 99 girls and 1 boy at WOCOMAL College. Not all of the students on campus are girls (only 98%), so the boy lives on campus. Therefore, there are 49 girls and 1 boy on campus, leaving $\boxed{50}$ students off campus.

2. Let $p = x + y$ and $q = xy$. Then, the first equation gives $p/q = 3/4$ and the second gives $pq = 48$. Therefore, $p^2 = 36$ so $p = \pm 6$ and $q = \pm 8$.

   For the positive case, $x + y = 6$ and $xy = 8$, so the possible values of x (and y, since the equations are symmetric) are the roots of the polynomial $a^2 - 6a + 8 = 0$, which are 2 and 4. Both are positive.

   For the negative case, $x + y = -6$ and $xy = -8$, so the possible values of x are the roots of the polynomial $a^2 + 6a - 8 = 0$, which the quadratic formula gives as $-3 \pm \sqrt{17}$. Since $\sqrt{17} > 4$, only $-3 + \sqrt{17}$ is positive.

   Therefore, the sum of all possible positive values of x is $2 + 4 + (-3 + \sqrt{17}) = \boxed{3 + \sqrt{17}}$.

3. Since x and y are restricted to the integers, let's try factoring the expression:

$$\begin{aligned} y^2 + 3x^2y^2 &= 30x^2 + 517 \\ y^2 + 3x^2y^2 - 30x^2 - 10 &= 517 - 10 \\ (y^2 - 10)(3x^2 + 1) &= 507 = 3 \cdot 13^2 \end{aligned}$$

   Since $x, y \in \mathbb{Z}$, each factor can be 1, 3, 13, 39, 169, or 507. The only combination that allows x and y to both be integers is $y^2 - 10 = 39$ and $3x^2 + 1 = 13$, so $(x^2, y^2) = (4, 49)$. Therefore, $3x^2y^2 = \boxed{588}$.

ROUND 3

1. To find the inverse function of $y = 2/x$, interchange x and y and solve for y. Then, $x = 2/y$ and $y = \boxed{2/x}$.

2. Plugging in, $f(g(x)) = 9x^2 + 9x + 11$ and $g(f(x)) = 3x^2 - 3x + 29$. Setting the two equal, we find $6x^2 + 12x - 18 = 0$ so $x = \boxed{-3, 1}$.

3. We are given that all coefficients and roots of f are integers and that $f(0) = -1$. Therefore, the product of all of the roots is -1 and so f has the form $a(x-1)^m(x+1)^n$. Plug in $x = 3$ to find $f(3) = a \cdot 2^m 4^n$. We wish to find the polynomial of least degree that satisfies the conditions, so we want to maximize n at the expense of m; this is achieved by letting $n = 3$ and $m = 1$ (such that $f(3) = \pm 128 = \pm 2^7$). From these values of m and n, we find that $a = 1$.

   Therefore, $f(x) = (x - 1)(x + 1)^3 = \boxed{x^4 + 2x^3 - 2x - 1}$.

ROUND 4

1. The total number of ways for five people to sit at a round table is $(5-1)! = 24$. Subtract off the number of ways that have Elizabeth and Lydia sitting next to each other, which is $2 \cdot (4-1)! = 12$. Our answer is therefore $24 - 12 = \boxed{12}$.

2. The key is to recognize the symmetry of the combinations function. The only way that the equality can be satisfied is if $(5x-13)+(8x-27) = x^2$. Solving, $x^2 - 13 + 40 = 0$ so $x = \boxed{5, 8}$. We check that both of these values satisfy $n \geq r \geq 0$ in both combinatorial expressions.

3. Express the position of each hand as a unit complex number. Then, for any valid point in time, $m = h^{12}$. We want the *interchanged* hand position to also be a valid point in time, so $h = m^{12} = h^{144}$. Therefore, $h^{143} = 1$ and the required points in time are when the hour hand points at the $\boxed{143}$ 143^{rd} roots of unity.

   (N.B. Strictly speaking, the complex plane starts with $0°$ as the "3 o'clock position" and θ moves in the counterclockwise direction. However, this is an easy bijection to make.)

ROUND 5

1. By definition, an ellipse is the locus of points where the sum of the distances to the two foci is constant. In this problem, that distance is $13 + 17 = 30$. Therefore, the length of the major axis is 30. To solve for the length of the minor axis, note that the center of the ellipse is the midpoint of the line connecting the two foci; here, that is $(10, 10)$. The distance from the center to a focus is $\sqrt{29}$, so the length of the minor axis is $2(\sqrt{15^2 - 29}) = 2 \cdot 14 = 28$. Therefore, the area of the ellipse is $\pi \cdot 15 \cdot 14 = \boxed{210\pi}$.

   [SIDE-NOTE: Although it is easy to find the *area* of the ellipse, it takes significantly more computation to find the *equation* describing the curve. This is an example of an OBLIQUE ELLIPSE, where the major and minor axes are not along the x- and y-axes.

   To find the equation, we must rotate everything using a standard rotation transformation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

   with θ chosen to be $\arctan(2/5)$. Then, from the Pythagorean Theorem, $\sin\theta = 2/\sqrt{29}$ and $\cos\theta = 5/\sqrt{29}$. After lots of algebra, the equation of the ellipse, in standard form, is $221x^2 + 20xy + 200y^2 - 4620x - 4200y = 0$. Note the presence of the xy term: a nonzero xy term indicates that the conic section's axes do not line up with the Cartesian axes.]

2. **METHOD I:** The shortest line connecting the point $(2, -3)$ to $y = (3x+1)/4$ will be perpendicular, so the slope is $-4/3$. A line with that slope passing through $(2, -3)$ has equation $y = -\frac{4}{3}x - \frac{1}{3}$. The intersection of the perpendicular line and original line can be found by setting those two equations equal; the intersection is at $(x, y) = (-7/25, 1/25)$.

   Finally, the required distance is $D = \sqrt{\left(\frac{57}{25}\right)^2 + \left(\frac{76}{25}\right)^2} = \frac{95}{25} = \boxed{\frac{19}{5}}$.

   METHOD II: If you happen to have the formula memorized, the distance from a point (m, n) to a line $Ax + By + C = 0$ is

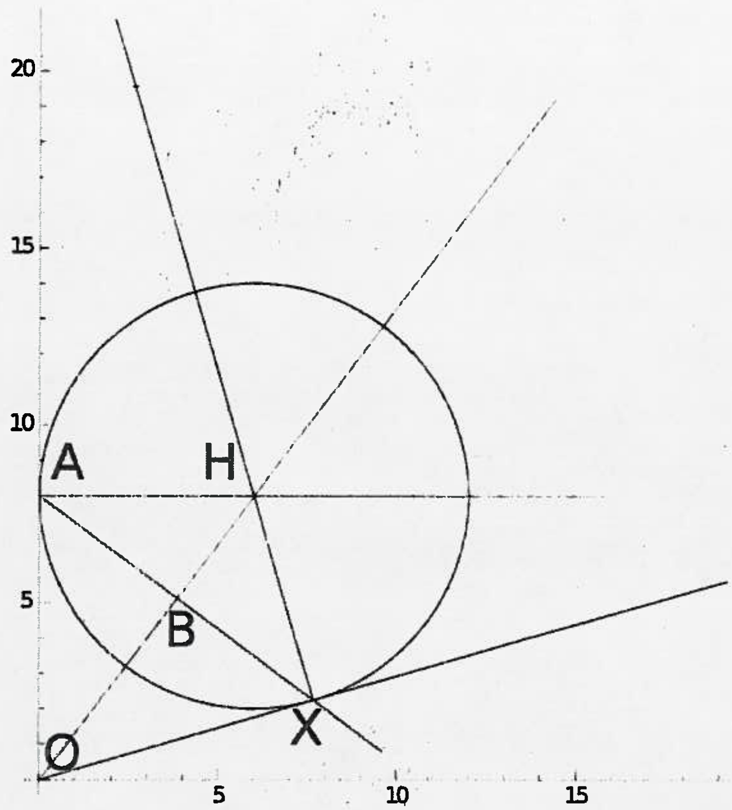
$$\frac{|Am + Bn + C|}{\sqrt{A^2 + B^2}}.$$

   Plugging in, $D = \dfrac{|3 \cdot 2 + 4 \cdot 3 + 1|}{\sqrt{3^2 + 4^2}} = \boxed{\frac{19}{5}}$.

3. Complete the square on the equation of the curve to find that it is a circle:

$$(x-6)^2 + (y-10)^2 = 36.$$

   The circle is centered at $(6, 10)$ with radius 6. The non-vertical line has y-intercept 2, so shift everything down 2 units so that the line passes through the origin and the circle of radius 6 is centered at $(6, 8)$:



METHOD I: The slope of $\overline{OH}$ is $4/3$, so the slope of $\overline{AX}$ is $-3/4$. By the Pythagorean Theorem, $OH = 10$ and $AX = 9.6$. We know the coordinates of A as $(0, 8)$, so using the slope and length of AX, we find that $X = (7.68, 2.24)$. Therefore, the slope of $\overline{OX}$ is $\frac{2.24}{7.68} = \boxed{\frac{7}{24}}$.

METHOD II: We know that triangles AOH and XOH are congruent by SSS, so $m\angle AOH = m\angle XOH = \theta$, with $\cot\theta = 4/3$. The problem then becomes finding the value of $\cot(2 \cdot \text{arccot}\,[4/3])$. This is nice if you know angle addition formulas for cotangent (I don't), but if you reflect everything across the line $y = x$, the cotangents become tangents with the slopes changed to their inverses.

Using this method, we have

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{2 \cdot (3/4)}{1 - (9/16)} = \frac{24}{7}.$$

Flip this back over the line $y = x$ to get that the slope is $\boxed{\frac{7}{24}}$.

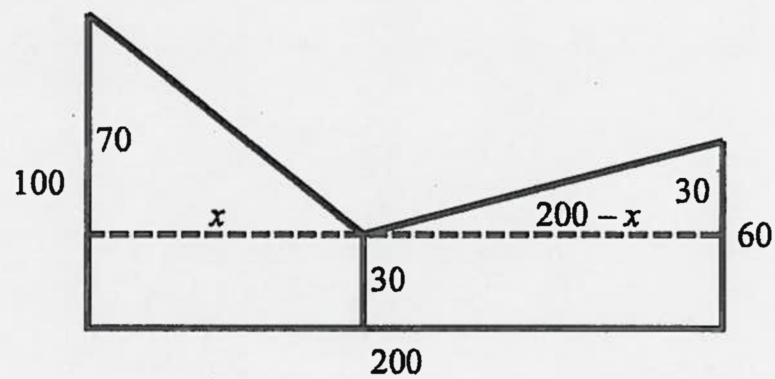
TEAM ROUND

1. **METHOD I:** The price of the discounted item is $x \cdot \left(1 - \frac{x}{100}\right) = x - x^2/100$. If we wish to maximize this, we can multiply through by -100 to get an expression to minimize: $x^2 - 100x$. Complete the square to get $(x-50)^2 - 2500$, and this expression is clearly at its minimum when $x = 50$ (squared quantities are non-negative).

   Therefore, the discounted price of the item is $\$50 \cdot \left(1 - \frac{50}{100}\right) = \boxed{\$25.00}$.

   METHOD II: Using calculus, we find that the derivative of $x - x^2/100$ is $1 - x/50$, so setting this to zero, we find that $x = 50$, as before.

2. Let x be the distance from the 100-foot pole.



   Then, by the Pythagorean Theorem, $70^2 + x^2 = (200-x)^2 + 30^2$ and $x = \boxed{90}$.

3. We use the "Stars and Bars" method of counting. Represent the ten candies with ten stars and use four bars to divide the selection into the five flavors. A possible configuration is

$$\star\star|\star\star\star||\star\star\star\star\star|$$

   which represents 2 of the first flavor, 3 of the second, 0 of the third, 5 of the fourth, and 0 of the fifth. Another allowed configuration is

$$\star\star|\star|\star\star|\star|\star\star\star\star$$

   which represents 2 of the first flavor, 1 of the second, 2 of the third, 1 of the fourth, and 4 of the fifth. The problem then becomes a question of how many ways there are to arrange the four bars among 14 objects (stars and bars combined). This is just $\binom{14}{4} = \boxed{1001}$.

4. It is easiest just to work backwards, from the outside in. The composite function consists of something minus 7; therefore, j must be the outside function. Next, we are left with a reciprocal, so that is $h(x) = 1/x$. After removing that, we have a polynomial in $\sin(x-7)$, so that is $g(x)$. Then we have $\sin(x-7)$, so that is $f(x)$. Finally, we are left with $x-7$, which is $j(x)$.

   Putting it all together, this combination is $\boxed{j(h(g(f(j(x)))))}$.

5. The line must be perpendicular to the line connecting the two points, so the slope is $-5/3$. The midpoint of the two points is $(1, 1)$, so the required line has equation $\boxed{y = -(5/3)x + 8/3}$.

6. By symmetry, every 6 consecutive n (giving rise to period 2π) sum to zero. Therefore, we choose to have $n = 15$ through 10 cancel and $n = 9$ through 4 cancel. The sum is then equal to just the case with $n = 3$, so this is $\cos\pi + i\sin\pi$. Since $\cos\pi = -1$ and $\sin\pi = 0$, we have $(a, b) = \boxed{(-1, 0)}$.

7. By VIETA'S FORMULAS, we have that $r + s = -b/a$ and $rs = c/a$. Then, we can rearrange:
$$\frac{1}{r^2} + \frac{1}{s^2} = \frac{r^2+s^2}{r^2s^2} = \frac{(r+s)^2 - 2rs}{(rs)^2}.$$
   Plugging in, this is equal to
$$\frac{(-b/a)^2 - 2(c/a)}{(c^2/a^2)} = \frac{(b^2/a^2) - (2ac/a^2)}{(c^2/a^2)} = \boxed{\frac{b^2-2ac}{c^2}}.$$

8. **METHOD I:** You can assume that the blue chip was drawn. This leaves 7 red and 2 blue chips in Bowl A before the transfer. Then, just find the probability that the four other transferred chips were 3 red and 1 blue, which is
$$\frac{\binom{7}{3}\binom{2}{1}}{\binom{9}{4}} = \frac{35\cdot 2}{126} = \boxed{\frac{5}{9}}.$$

   METHOD II: We know that a blue chip was drawn from Bowl B, so the only possibilities for the initial transfer are: i) 2 red and 3 blue; ii) 3 red and 2 blue; or iii) 4 red and 1 blue. The number of ways that each scenario can happen can be calculated using combinations: i) $\binom{7}{2}\binom{3}{3} = 21$; ii) $\binom{7}{3}\binom{3}{2} = 105$; and iii) $\binom{7}{4}\binom{3}{1} = 105$. However, we must weight these in a $3:2:1$ ratio because of the relative probabilities of drawing a blue chip from each of these subsets of 5 chips.

The problem asks for the conditional probability that the transferred chips were 3 red and 2 blue, so the required weighted average is

$$\frac{2 \cdot 105}{3 \cdot 21 + 2 \cdot 105 + 1 \cdot 105} = \boxed{\frac{5}{9}}.$$

9. We have

$$\begin{aligned} \frac{\frac{0.\overline{3} - 0.3\overline{9}}{0.\overline{3} + 0.\overline{4}}}{3.\overline{9}} &= \frac{\frac{0.\overline{3} - 0.4}{0.\overline{7}}}{4} \\ &= \left(\frac{1}{3} - \frac{2}{5}\right) \cdot \frac{9}{7} \cdot \frac{1}{4} \\ &= \frac{-1}{15} \cdot \frac{9}{7} \cdot \frac{1}{4} \\ &= \boxed{-\frac{3}{140}}. \end{aligned}$$